18.06 MIDTERM 2

November 1, 2019 (50 minutes)

Please turn cell phones off completely and put them away.

No books, notes, or electronic devices are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Please write your name on **ALL** pages that you want graded (those will be the ones we scan).

The back sides of the paper will **NOT** be graded (for scratch work only).

Do not unstaple the exam, nor reorder the sheets.

Problem 1 has $\boxed{5}$ parts, Problem 2 has $\boxed{3}$ parts, Problem 3 has $\boxed{3}$ parts.

NAME: Sveta (~25 min)

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SCRATCH WORK

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PROBLEM 1 **NAME:** ____________________

(1) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be linearly independent vectors in $\mathbb{R}^n$. What is the rank of the matrix:

$$A = [\mathbf{v}_1|\mathbf{v}_2]$$

whose columns are the given vectors? *(5 pts)*

Do they really need to justify this?

(2) For any vector $\mathbf{b} \in \mathbb{R}^n$, its projection onto the subspace V spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is:

$$\text{proj}_V \mathbf{b} = P_V \mathbf{b} \qquad \text{where the projection matrix is} \qquad \boxed{P_V = A(A^T A)^{-1} A^T}$$

Use this to obtain a formula (in terms of A and $\mathbf{b}$) for the real numbers α and β defined by the property that $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ is the closest vector in the subspace V to the vector $\mathbf{b}$. *(10 pts)*

$$\alpha v_1 + \beta v_2 = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = Pb$$

$$A^T A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A^T P b = A^T b$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (A^T A)^{-1} A^T b$$

Might be hard, b/c not everybody is very comfortable w/ the idea

$$\alpha v_1 + \beta v_2 = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

SCRATCH WORK

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NAME: ______________________

(3) Explain why, for any given vector $\mathbf{b}$, the numbers α and β in (2) are unique. *(5 pts)*

Mb comment what kind of explanation we expect?

(4) Let $\mathbf{v}_1$, $\mathbf{v}_2$, A still be as on the previous page, and consider the vectors $\mathbf{w}_1 = 3\mathbf{v}_1 - 2\mathbf{v}_2$ and $\mathbf{w}_2 = 2\mathbf{v}_1 - \mathbf{v}_2$. We consider the matrix whose columns are these new vectors:

$$B = \left[\mathbf{w}_1 | \mathbf{w}_2\right]$$

Decide whether $B = AX$ <u>or</u> $B = XA$ for some matrix X. What is X? <u>Explain</u>. *(5 pts)*

$$B = \left[\, 3v_1 - 2v_2 \;\middle|\; 2v_1 - v_2 \,\right] =$$

$$= \left[\, A\begin{bmatrix} 3 \\ -2 \end{bmatrix} \;\middle|\; A\begin{bmatrix} 2 \\ -1 \end{bmatrix} \,\right] =$$

$$= A\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

SCRATCH WORK

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NAME: ____________________

(5) The projection matrix onto the subspace W spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ is:

$$\boxed{P_W = B(B^T B)^{-1} B^T}$$

Prove that $P_W = P_V$, <u>either</u> by a geometric argument, <u>or</u> by a computation. *(10 pts)*

$$AX\left(X^T A^T A X\right)^{-1} X^T A^T =$$

$$= A X X^{-1} (A^T A) X^{T,-1} X^T A^T$$

SCRATCH WORK

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PROBLEM 2 **NAME:** ____________________

(1) Use Gram-Schmidt to obtain a factorization (show all your steps):

$$\boxed{A = QR} \qquad \text{of the matrix} \qquad A = \begin{bmatrix} 1 & 6 \\ 4 & 15 \\ 8 & 12 \end{bmatrix}$$

where Q has orthonormal columns and R is an upper triangular square matrix. *(15 pts)*

$$\begin{bmatrix} 1 & 6 \\ 4 & 15 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 1/9 & 6 \\ 4/9 & 15 \\ 8/9 & 12 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/9 & 4 \\ 4/9 & 7 \\ 8/9 & -4 \end{bmatrix} \begin{bmatrix} 1 & 18 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} = \star$$

$$w - \frac{(v,w)}{(v,v)} v = \begin{bmatrix} 6 \\ 15 \\ 12 \end{bmatrix} - \frac{2+20+32}{3} \begin{bmatrix} 1/9 \\ 4/9 \\ 8/9 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 15 \\ 12 \end{bmatrix} - \frac{54}{3} \begin{bmatrix} 1/9 \\ 4/9 \\ 8/9 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 12 \end{bmatrix} - \begin{bmatrix} 2 \\ 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -4 \end{bmatrix}$$

$$\text{len} = \sqrt{32+49} = \sqrt{81} = 9$$

$$\star = \begin{bmatrix} 1/9 & 4/9 \\ 4/9 & 7/9 \\ 8/9 & -4/9 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 9 & 18 \\ 0 & 1 \end{bmatrix}}_{R = \begin{bmatrix} 9 & 18 \\ 0 & 9 \end{bmatrix}}$$

SCRATCH WORK

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NAME: ____________________

(2) With the notation as on the previous page, consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(\mathbf{v}) = Q\mathbf{v}$$

Suppose you have any two orthogonal (i.e. perpendicular) vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Prove that the vectors $f(\mathbf{v}_1), f(\mathbf{v}_2) \in \mathbb{R}^3$ are also orthogonal (justify all your steps). *(10 pts)*

Do we want to keep the $Q = \begin{bmatrix} 1/9 & \cdots \\ 4/9 & \\ \cdots & \end{bmatrix}$?

$$f(v_1)^T f(v_2) = v_1^T Q^T Q v_2 =$$

$$= v_1^T v_2 = 0$$

SCRATCH WORK

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NAME: ____________________

(3) Compute an eigenvector $\mathbf{a}$ of the matrix R and the corresponding eigenvalue *(Hint: it's easy to spot the eigenvector just by looking at the matrix R).* Draw the linear transformation:

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad g(\mathbf{w}) = R\mathbf{w}$$

on a picture of $\mathbb{R}^2$, by drawing the eigenvector $\mathbf{a}$ and showing where the function g sends $\mathbf{a}$ and any other vector in $\mathbb{R}^2$ of your choice, linearly independent from $\mathbf{a}$. *(10 pts)*

SCRATCH WORK

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PROBLEM 3 **NAME:** ____________________

(1) Assume a, d, f are non-zero numbers and b, c, e are arbitrary. Compute all 9 cofactors of:

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

and use them to obtain a formula for the inverse matrix A^{-1}. You may use the well-known formula for 2×2 determinants $\det \begin{bmatrix} x & y \\ z & t \end{bmatrix} = xt - yz$. *(10 pts)*

$$\frac{1}{adf} \begin{bmatrix} df & -bf & be-cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix}$$

SCRATCH WORK

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NAME: ____________________

(2) Explain why all $5! = 120$ terms in the big formula for the determinant:

$$\begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

are zero. *(10 pts)*

SCRATCH WORK

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NAME: ____________________

(3) Use row operations to compute the determinant of the matrix:

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{bmatrix}$$

(if instead of row operations, you use the formula for 3×3 determinants as a sum of 6 terms to compute the above, you will lose at least half of the points). *(10 pts)*

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 \\ 0 & \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 \\ 0 & 0 & \gamma^2-\alpha^2 - (\beta^2-\alpha^2)\frac{\gamma-\alpha}{\beta-\alpha} \end{vmatrix}$$

$$= (\beta-\alpha)\Big((\gamma-\alpha)(\gamma+\alpha) - (\beta+\alpha)(\gamma-\alpha)\Big) =$$

$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)$$

Should it be pointed out that α, β, γ can be equal?

SCRATCH WORK

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